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On the decay of weak shock waves in axisymmetric non-equilibrium flow

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The supersonic flight of a slender projectile in a fluid which may undergo internal non-equilibrium transformations is examined by a systematic perturbation scheme. In the frozen limit, the classical results of Whitham and his celebrated 'rule' are recovered. Unlike the classical theory, however, the shape of the nose shock can be expressed explicitly in terms of known functions when the relaxation decay length becomes of the same order as a characteristic length scale. The theoretically predicted shock angle, expressed as a function of the radial distance, is found to be in excellent agreement with the experimental measurements of Wegener and Klikoff.

1. Introduction

Theoretical and experimental investigation of the decay of weak conical shock waves in a non-equilibrium flow was described in an earlier paper (Wegener, Chu & Klikoff 1965). Though adequate for interpreting the experimental results, the theory developed there is deficient in many respects. While it accounted satisfactorily for both the dispersion and geometrical effects on the decay of conical shocks, it ignored entirely the nonlinear effects. In particular, the theory failed to give an expression for the variation of the shock angle δ with the radial distance r.

The purpose of the present study is to remedy this deficiency. A perturbation theory is formulated which allows one to predict the shock shape to any degree of accuracy. The theory, calculated to the second order of the semi-vertex angle of the projectile, recovers in the limiting case of a frozen flow Whitham's (1952) theory of a supersonic projectile. In general, the shock shape cannot be expressed explicitly in terms of known functions; it is given implicitly in parametric form as in classical theory. However, in the important special case where the relaxation decay length κ^{-1} (defined by (3.25) below) is of the order of a characteristic length of the projectile, the nose shock assumes the simple form

$$x = \lambda_0 r - \tfrac{3}{16}\pi k^2 \epsilon^4 \kappa^{-1}\{\operatorname{erf}(kr)^{\frac{1}{2}}\}^2, \tag{1.1}$$

where $\lambda_0 = (M_{f0}^2 - 1)^{\frac{1}{2}}$, M_{f0} being the frozen Mach number, ϵ is effectively the semi-vertex angle and k is a dimensionless parameter defined by (4.9). The extraordinary simplicity of this expression is a consequence of and reflection on the physical fact that only Mach waves near the very tip of the projectile overtake the nose shock, attenuating it to a vanishing weak discontinuity in a few relaxation decay lengths. It is also for this reason that ϵ is the only geometrical parameter which enters into the formula. The shock angles computed from the above equation are found to be in excellent agreement with the experimental results of Wegener and Klikoff.

2. Description of the problem

The basic equations governing the motion of a steady axisymmetric non-equilibrium supersonic flow over a projectile are well known. Neglecting the various transport effects and assuming only one non-equilibrium mode, they may be written in the form

$$\left.\begin{aligned} &\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}\right)+\frac{\partial p}{\partial x}=0, \quad \rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial r}\right)+\frac{\partial p}{\partial r}=0,\\ &\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial r}+\frac{\rho v}{r}=0, \quad u\frac{\partial q}{\partial x}+v\frac{\partial q}{\partial r}=\dot{q}(p,\rho,q),\\ &\frac{1}{a_f^2}\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial r}\right)-\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial r}\right)=\frac{h_q}{h_\rho}\dot{q}, \quad h=h(p,\rho,q).\end{aligned}\right\} \tag{2.1}$$

Here, x is the distance along the axis of the symmetry measured from the tip of the projectile in the direction of the oncoming flow, and r the radial distance from the axis; u and v denote respectively the velocity components along the x and r axis; p, ρ, h, q and $\dot{q}$ are respectively the pressure, density, specific enthalpy, a variable characterizing the progress of the non-equilibrium process and the reaction rate (describing the rate of change of the progress variable q);

$$\dot{q} \equiv \dot{q}(p,\rho,q)$$

is a known function of p, ρ and q. Finally,

$$h_q = \partial\{h(p,\rho,q)\}/\partial q,\ h_\rho = \partial h/\partial \rho,\ h_p = \partial h/\partial p$$

and the frozen sound speed a_f is given by

$$a_f^2 = (\partial p/\partial \rho)_{s,q} = -h_\rho/(h_p - 1/\rho), \tag{2.2}$$

(cf. Vincenti & Kruger 1965). When the frozen Mach number $M_f = [(u^2+v^2)/a_f^2]^{\frac{1}{2}}$ is greater than one, the system (2.1) possesses three families of real characteristics: the outgoing and incoming Mach waves, and the streamlines. In terms of u, v and a_f, an outgoing Mach wave has a slope determined by

$$\frac{dx}{dr} = \frac{-uv + a_f(u^2+v^2-a_f^2)^{\frac{1}{2}}}{a_f^2 - v^2} = \lambda, \tag{2.3}$$

where $\lambda = \cot(\theta + \mu_f)$, $\theta = \tan^{-1}(v/u)$ being flow angle and μ_f being the frozen Mach angle.

In studying the shape of the nose shock, it is essential (see e.g. Chu 1970) to introduce a new co-ordinate system (α, β) defined as follows: α is constant along an outgoing Mach line such that if this line intersects the surface of the projectile at a point $x = x^*$, the line will be labelled as $\alpha = x^*$; β is simply r. It follows that in the region of the flow field affected by the projectile motion $x = x(\alpha, \beta)$ and $r = \beta$. The transformation relationships between (x, r) on the one hand and (α, β) on the other can be deduced immediately from $dx = x_\alpha d\alpha + x_\beta d\beta$ where subscripts α and β signify partial differentiation with respect to α and β respectively. For example, $\partial\alpha/\partial x = 1/x_\alpha$, $\partial\alpha/\partial r = -x_\beta/x_\alpha$, etc. In terms of α and β, (2.1) and (2.3) become

$$\left.\begin{aligned} \rho\{(u-\lambda v)u_\alpha + vu_\beta x_\alpha\} = -p_\alpha,\ \rho\{(u-\lambda v)v_\alpha + vv_\beta x_\alpha\} = \lambda p_\alpha - p_\beta x_\alpha, \\ (\rho u)_\alpha - \lambda(\rho v)_\alpha + x_\alpha(\rho v)_\beta = -\rho v x_\alpha/\beta,\ (u-\lambda v)q_\alpha + vq_\beta x_\alpha = \dot{q}x_\alpha, \\ (1/a_f^2)\{(u-\lambda v)p_\alpha + vp_\beta x_\alpha\} - \{(u-\lambda v)\rho_\alpha + v\rho_\beta x_\alpha\} = (h_q/h_\rho)\dot{q}x_\alpha,\ x_\beta = \lambda. \end{aligned}\right\} \quad (2.4a\text{–}f)$$

Let the projectile be described by the equation $r = \epsilon R(x)$ where ϵ may be taken as the slope dr/dx of the projectile at $x = 0$. We shall be concerned here with slender and smooth projectiles. That is, ϵ is assumed small and $R(x)$ is a differentiable function for $x > 0$. Generalization to cases where dr/dx may have jump discontinuities along the projectile presents no basic difficulties.

The requirement that the flow should be tangentral to the projectile implies that $v/u \to \epsilon R'(x)$ as $r \to \epsilon R(x)$. In the α, β plane, we have

$$\left.\begin{aligned} v/u \to \epsilon R'(\alpha) \quad \text{as} \quad \beta \to \epsilon R(\alpha), \\ x = a \quad \text{at} \quad \beta = \epsilon R(\alpha). \end{aligned}\right\} \quad (2.5a, b)$$

The last condition follows directly from the particular way in which the characteristics are labelled. In addition to these boundary conditions, the usual jump conditions must be satisfied at each point on the nose shock. Thus, if δ is the shock angle and subscript 0 denotes the free-stream condition, which is assumed to be in thermodynamic equilibrium, we have

$$\left.\begin{aligned} \rho(u - v\cot\delta) = \rho_0 u_0, \quad h + \tfrac{1}{2}u^2 = h_0 + \tfrac{1}{2}u_0^2, \quad q = q_0, \\ p - p_0 = \rho_0 u_0(u_0 - u), \qquad v = (u_0 - u)\cot\delta, \end{aligned}\right\} \quad (2.6)$$

representing respectively the continuity equation, the energy equation, the continuity of q, and the momentum balance in the normal and tangential directions. The position of the nose shock is, of course, not known *a priori*. It must assume a form so that $dx/dr = \cot\delta$ at every point on the nose shock. In the α, β plane, the system of boundary conditions (2.6) must be applied along the shock locus $\alpha = \alpha(\beta, \epsilon)$ determined by integrating

$$x_\alpha d\alpha/d\beta + x_\beta = \cot\delta, \quad \text{or} \quad d\alpha/d\beta = (\cot\delta - x_\beta)/x_\alpha, \quad (2.7)$$

subject to the initial condition $\alpha = 0$ at $\beta = 0$.

Obviously the problem is a very complicated one. Nevertheless, as will be seen, a systematic calculation of the nose shock can be carried out to any degree of accuracy for the case of a slender body.

3. Solution

3.1. *Perturbation equations*

Let $\psi(\alpha,\beta)$ be a dependent variable which appears in the system of equations (2.4). We assume that it can be expressed as

$$\psi(\alpha,\beta) = \psi_0(\alpha,\beta)+\epsilon\psi_1(\alpha,\beta)+\epsilon^2\psi_2(\alpha,\beta)+\dots . \tag{3.1}$$

Obviously $\psi = \psi_0$ for $\epsilon = 0$ so that ψ_0 is the value of ψ for a uniform stream. In particular, p_0, ρ_0, q_0 and u_0 are respectively the free-stream pressure, density, progress variable and velocity; moreover $v_0 = 0$. Writing the dependent variables of the system (2.4) in the form (3.1) and collecting terms of like order in ϵ, we obtain a set of equations governing terms of zeroth-, first-, second-, ... order variables. For example, (2.4*f*) gives

$$x_{0\beta} = \lambda_0, \quad x_{1\beta} = \lambda_1 = -M_{f0}^2\frac{v_1}{u_0}+\frac{M_{f0}^2}{\lambda_0}\left(\frac{u_1}{u_0}-\frac{a_1}{a_{f0}}\right), \dots, \tag{3.2a, b}$$

where

$$\left.\begin{aligned} \lambda_0 &= (M_{f0}^2-1)^{\frac{1}{2}} = \cot\mu_{f0}, \quad a_f = a_{f0}+\epsilon a_1+\dots, \\ a_1 &= \left(\frac{\partial a_f}{\partial p}\right)_0 p_1+\left(\frac{\partial a_f}{\partial \rho}\right)_0 \rho_1+\left(\frac{\partial a_f}{\partial q}\right)_0 q_1. \end{aligned}\right\} \tag{3.3}$$

Likewise, (2.4*a*) to (2.4*e*) give

$$\left.\begin{aligned} &\rho_0 u_0 u_{1\alpha} = -p_{1\alpha}, \quad \rho_0 u_0 v_{1\alpha} = \lambda_0 p_{1\alpha}-p_{1\beta}x_{0\alpha}, \\ &\rho_0 u_{1\alpha}+\rho_{1\alpha}u_0-\lambda_0\rho_0 v_{1\alpha}+x_{0\alpha}\rho_0 v_{1\beta} = -\rho_0 v_1 x_{0\alpha}/\beta, \; u_0 q_{1\alpha} = \dot{q}_1 x_{0\alpha}, \\ &\frac{1}{a_{f0}^2}p_{1\alpha}-\rho_{1\alpha} = \frac{h_{q0}}{h_{\rho 0}}\frac{x_{0\alpha}}{u_0}\dot{q}_1, \end{aligned}\right\} \tag{3.4}$$

etc. where $\quad \dot{q}_1 = \dot{q}_{q0}(q_1-q_1^*), \; q_1^* = (\partial q^*/\partial p)_0 p_1+(\partial q^*/\partial\rho)_0\rho_1, \qquad (3.5)$

and $q^* = q^*(p,\rho)$ is the equilibrium value of the progress variable q at a pressure p and density ρ.

In a similar manner, the boundary condition (2.5) can be decomposed into a set of conditions, according to different powers of ϵ, with the help of Taylor's expansion. For example, (2.5*b*) gives

$$\left.\begin{aligned} &x_0(\alpha,0) = \alpha, \quad x_1(\alpha,0)+x_{0\beta}(\alpha,0)R(\alpha) = 0, \\ &x_2(\alpha,0)+x_{1\beta}(\alpha,0)R(\alpha)+\tfrac{1}{2}x_{0\beta\beta}(\alpha,0)R(\alpha)^2 = 0, \dots . \end{aligned}\right\} \tag{3.6a, b, c}$$

However, care should be exercised in applying the same procedure to (2.5*a*) for it is well known and easily demonstrable that $v \sim 1/r$ or $1/\beta$ as $\beta\to 0$ so that $v(\alpha,\beta)$ does not possess a Taylor expansion at $\beta = 0$. The difficulty can be circumvented by writing $v = V(\alpha,\beta)/\beta$ where $V(\alpha,\beta)$ has a Taylor expansion at $\beta = 0$. In this way one deduces from (2.5*a*) that

$$\lim_{\beta\to 0}\beta v_1 = 0, \quad \lim_{\beta\to 0}\{\beta v_2+R(\alpha)d(\beta v_1)/d\beta\} = u_0 R(\alpha)R'(\alpha), \dots . \tag{3.7a, b}$$

3.2. *Conditions at the nose shock*

Substituting expressions like (3.1) into the jump conditions (2.6) and collecting terms of like power in ϵ, we obtain a set of boundary conditions at the shock wave.

The first-order boundary conditions are

$$\left.\begin{aligned} &\rho_0(u_1 - v_1\cot\delta_0) + \rho_1 u_0 = 0, \quad h_{p0}p_1 + h_{\rho 0}\rho_1 + h_{q0}q_1 + u_0 u_1 = 0, \\ &q_1 = 0, \quad p_1 + \rho_0 u_0 u_1 = 0, \quad v_1 + u_1 \cot\delta_0 = 0. \end{aligned}\right\} \tag{3.8}$$

Unlike (2.6), this is a homogeneous system in p_1, ρ_1, u_1, v_1 and q_1. If these are not simultaneously zero at the shock, the determinant of the system must vanish. It follows that $\cot\delta_0 = (M_{f0}^2 - 1)^{\frac{1}{2}} = \lambda_0$ or $\delta_0 = \mu_{f0}$, the free-stream frozen Mach angle. Moreover,

$$p_1 = a_{f0}^2\rho_1 = -\rho_0 u_0 u_1 = (\rho_0 u_0/\lambda_0)v_1, \quad q_1 = 0 \tag{3.9}$$

at the shock. Likewise, collecting terms of ϵ^2, one obtains

$$\left.\begin{aligned} &\rho_0(u_2 - v_2\cot\delta_0) + \rho_2 u_0 = -\rho_0 M_{f0}^2\delta_1 v_1 + \rho_0 u_0 (M_{f0}^2/\lambda_0^2)v_1^2/a_{f0}^2, \\ &h_{p0}p_2 + h_{\rho 0}\rho_2 + h_{q0}q_2 + u_0 u_2 = -\tfrac{1}{2}(M_{f0}^2/\lambda_0^2)v_1 \\ &\qquad\qquad \times \{1 + \rho_0^2 a_{f0}^2 (h_{pp} + (2/a_f^2)h_{p\rho} + (1/a_f^4)h_{\rho\rho})_0\}, \\ &q_2 = 0, \quad p_2 + \rho_0 u_0 u_2 = 0, \quad v_2 + u_2\cot\delta_0 = -M_{f0}^2\delta_1 v_1/\lambda_0, \end{aligned}\right\} \tag{3.10}$$

where subscript 0 appended to a bracket signifies that all quantities in the bracket are evaluated at the undisturbed state of the oncoming free stream. In deriving (3.10) use has been made of (3.9) in simplifying the first-order terms. Now the homogeneous part of (3.10) and (3.8) are the same; in particular, their determinants are zero when $\delta_0 = \mu_f$. To ensure the system that (3.10) is consistent, its non-homogenous terms must be related appropriately. This relationship may be easily obtained by eliminating the second-order quantities from the system (3.10). Thus,

$$\delta_1 = \tfrac{1}{4}(A+1)(M_{f0}^2/\lambda_0^2)v_1/u_0, \tag{3.11}$$

where

$$A = 1 + \frac{a_{f0}^2}{\rho_0 h_{\rho 0}}\left\{1 + \rho^2 a_f^2\left(h_{pp} + \frac{2}{a_f^2}h_{p\rho} + \frac{1}{a_f^4}h_{\rho\rho}\right)\right\}_0. \tag{3.12}$$

For an ideal gas, $h = (\gamma/(\gamma-1))p/\rho$ and $a_f^2 = \gamma p/\rho$ so that A is just γ. Since

$$\cot\delta = \cot(\delta_0 + \epsilon\delta_1 + \ldots) \quad \text{and} \quad \cot\delta_0 = \cot\mu_{f0} = \lambda_0,$$

$$\cot\delta = \lambda_0 - \epsilon M_{f0}^2\delta_1 + O(\epsilon^2) = \lambda_0 - \tfrac{1}{4}(A+1)\epsilon(M_{f0}^4/\lambda_0^2)v_1/u_1 + O(\epsilon^2). \tag{3.13}$$

3.3. *Zeroth- and first-order solutions*

The zeroth- and first-order solutions can be written down by inspection. Thus, it is obvious from (3.2*a*) and (3.6*a*) that

$$x_0 = \lambda_0\beta + \alpha. \tag{3.14}$$

On the other hand, the system of equations (3.4), (3.7) and (3.9) contains no inhomogeneous terms so that

$$p_1 = 0, \quad \rho_1 = 0, \quad q_1 = 0, \quad u_1 = 0, \quad v_1 = 0. \tag{3.15}$$

The only first-order quantity that is non-zero is $x_1(\alpha, \beta)$ which, according to (3.2*b*) and (3.6*b*), is given by

$$x_1 = -\lambda_0 R(\alpha). \tag{3.16}$$

3.4. *Second-order solution*

In view of (3.15), the second-order quantities, $p_2, \rho_2, q_2, u_2, v_2$ and x_2, satisfy the same set of differential equations as $p_1, \rho_1, q_1, u_1, v_1$ and x_1. Likewise, the second-order boundary conditions at the shock are given by (3.9) with subscript 1 replaced by 2. Similarly, (3.11) and (3.13) become

$$\delta_2 = \tfrac{1}{4}(A+1)\frac{M_{f0}^2}{\lambda_0^2}\frac{v_2}{u_0}, \quad \cot\delta = \lambda_0 - \tfrac{1}{4}(A+1)\epsilon^2\frac{M_{f0}^4}{\lambda_0^2}\frac{v_2}{u_0}+\ldots, \qquad (3.17\,a,b)$$

which is now valid to the order of ϵ^2.

On the other hand, the second-order boundary conditions at the projectile are given by (3.6*c*) and (3.7*b*). The solution of this boundary-value problem can be readily constructed by the method of Laplace transform if the jump conditions at the shock could be applied at $\alpha = 0$ instead of the shock wave itself. Of course, there is no *a priori* assurance that such a replacement is legitimate and yields a solution which is, at most, different from the required solution to a higher order in ϵ. While it is true that the shock wave degenerates into the frozen Mach wave $\alpha = 0$ in the limit $\epsilon \to 0$, this feature alone is not sufficient to guarantee the validity of the replacement. At any rate the accuracy of such an approximation will have to be examined after the shock locus is computed (cf. §5).

Denoting the Laplace transform of a function $Q(\alpha,\beta)$ by $\bar{Q}(s,\beta)$ or simply $\bar{Q}$, i.e.

$$\bar{Q} = \int_0^\infty Q(\alpha,\beta)e^{-\alpha s}d\alpha, \qquad (3.18)$$

we find

$$\left.\begin{aligned}
\bar{u}_2 &= -(u_0/2\pi)s\bar{S}'K_0\{\omega(s)\beta\}e^{\lambda_0\beta s},\\
\bar{p}_2 &= -\rho_0 u_0 \bar{u}_2, \quad \bar{\rho}_2 = \frac{\bar{p}_2}{a_{f0}^2}\left(1+\frac{\sigma}{1+u_0\tau_0 s}\right),\\
\bar{q}_2 &= -\frac{h_{\rho 0}}{h_{q0}}\left(\bar{\rho}_2 - \frac{\bar{p}_2}{a_{f0}^2}\right),\\
\bar{v}_2 &= (u_0/2\pi)\omega(s)\bar{S}'K_1\{\omega(s)\beta\}e^{\lambda_0\beta s},
\end{aligned}\right\} \qquad (3.19\,a\text{–}e)$$

where $S = \pi R(\alpha)^2$ is the cross-sectional area of the projectile at α, $S' = dS/d\alpha$, and $\bar{S}'$ is the Laplace transform of S'. In the above solutions, K_0 and K_1 are modified Bessel functions of order zero and one respectively, and

$$\omega(s) = s\left[\lambda_0^2 + \frac{\sigma M_{f0}^2}{1+u_0\tau_0 s}\right]^{\frac{1}{2}}, \quad \sigma = \frac{a_{f0}^2}{a_{e0}^2} - 1, \quad \tau_0 = -\frac{1}{\dot{q}_{q0}}\frac{h_{\rho 0}}{h_{\rho 0}+h_{q0}q_{\rho 0}^*}, \qquad (3.20)$$

where a_e is the equilibrium sound speed and $a_e^2 = -(h_\rho + h_q q_\rho^*)/(h_p + h_q q_p^* - 1/\rho)$.

To complete the second-order solution in the transform plane, we must compute $x_2(s,\beta)$. This is found to be governed by the equation

$$\bar{x}_{2\beta} = -M_{f0}^2\frac{\bar{v}_2}{u_0} - \Lambda\frac{\bar{p}_2}{\rho_0 u_0^2}, \qquad (3.21)$$

and the boundary condition, $x_2(s,0) = 0$, where

$$\Lambda = \frac{M_{f0}^2}{\lambda_0}\left\{1+\rho_0 u_0 M_{f0}\left(\frac{\partial a_f}{\partial p}+\frac{1}{a_f^2}\frac{\partial a_f}{\partial \rho}\right)_0+\frac{\sigma}{1+u_0\tau_0 s}\frac{\rho_0 M_{f0}^2}{a_{f0}^2}\left(\frac{\partial a_f}{\partial \rho}-\frac{h_q}{h_\rho}\frac{\partial a_f}{\partial q}\right)_0\right\}. \quad (3.22)$$

3.5. *Behaviour of solution for small α*

The behaviour of the second-order solution for small α can be deduced from the nature of their transform at large s. Now, for $u_0\tau_0 s \gg 1$,

$$\omega(s) \sim s\lambda_0 + \frac{\sigma M_{f0}^2}{2\lambda_0}\frac{1}{u_0\tau_0} + O\left(\frac{1}{s}\right). \quad (3.23)$$

In addition, if $\lambda_0\beta s \gg 1$,

$$\kappa_\nu\{\omega(s)\beta\}e^{\lambda_0\beta s} \sim \left(\frac{\pi}{2\lambda_0\beta s}\right)^{\frac{1}{2}} e^{-\kappa\beta}, \quad (3.24)$$

for all values of ν, where $\kappa = (\sigma M_{f0}^2/2\lambda_0)/u_0\tau_0$. (3.25)

Consequently, for $s \gg 1/u_0\tau_0$ and $1/\lambda_0\beta$,

$$\left.\begin{aligned}
\bar{v}_2 &\sim \tfrac{1}{2}u_0\lambda_0\bar{S}'\left(\frac{s}{2\pi\lambda_0\beta}\right)^{\frac{1}{2}} e^{-\kappa\beta},\\
\bar{u}_2 &= -\bar{v}_2/\lambda_0, \quad \bar{p}_2 = -\rho_0 u_0 u_2, \quad \bar{p}_2 = a_{f0}^2\bar{\rho}_2,\\
\bar{q}_2 &\sim -\frac{\rho_0 h_{\rho 0}}{h_{q0}}\frac{\sigma M_{f0}^2}{2}\frac{1}{u_0\tau_0}\bar{S}'\left(\frac{1}{2\pi\lambda_0\beta s}\right)^{\frac{1}{2}} e^{-\kappa\beta},\\
\bar{x}_{2\beta} &= -\frac{A+1}{2}\frac{M_{f0}^4}{\lambda_0^2}\frac{\bar{v}_2}{u_0},
\end{aligned}\right\} \quad (3.26)$$

where

$$A = 1+\rho_0 a_{f0}\left(\frac{\partial a_f}{\partial p}+\frac{1}{a_f^2}\frac{\partial a_f}{\partial \rho}\right)_0. \quad (3.27)$$

Substituting (2.2) into (3.27), it is easy to verify that the A defined here is the same as that introduced earlier in (3.12).

The system of equations (3.26) may be readily inverted. Now the inverse Laplace transform of $(s/4\pi)^{\frac{1}{2}}\bar{S}'$ is

$$F(\alpha) = \frac{1}{2\pi}\int_0^\alpha \frac{S''(\xi)\,d\xi}{(\alpha-\xi)^{\frac{1}{2}}}. \quad (3.28)$$

Consequently, for $\alpha/u_0\tau_0$ and $\alpha/\lambda_0\beta \ll 1$,

$$\left.\begin{aligned}
v_2 &= u_0(\lambda_0/2\beta)^{\frac{1}{2}} e^{-\kappa\beta}F(\alpha),\\
u_2 &= -v_2/\lambda_0, \quad p_2 = -\rho_0 u_0 u_2 = a_{f0}^2\rho_2,\\
q_2 &= -\frac{\rho_0 h_{\rho 0}}{h_{q0}}\frac{\sigma M_{f0}^2}{\lambda_0}\frac{1}{u_0\tau_0}\left(\frac{\lambda_0}{2\beta}\right)^{\frac{1}{2}} e^{-\kappa\beta}\frac{1}{2\pi}\int_0^\alpha \frac{S'(\xi)\,d\xi}{(\alpha-\xi)^{\frac{1}{2}}},\\
x_2 &= -\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{A+1}{2}\frac{M_{f0}^4}{(\lambda_0^3\kappa)^{\frac{1}{2}}}\operatorname{erf}((\kappa\beta)^{\frac{1}{2}})F(\alpha),
\end{aligned}\right\} \quad (3.29\,a\text{–}e)$$

where use is made of the boundary condition $x_2(\alpha, 0) = 0$. Evidently $1/\kappa$ is the attenuation or decay length due to non-equilibrium effects in the fluid.

4. Nose shock

4.1. *Shock locus*

According to (2.7), the nose shock in the α, β plane is

$$d\alpha/d\beta = (\cot\delta - x_\beta)/x_\alpha, \tag{4.1}$$

where $\cot\delta$ is given by (3.17*b*). Since the nose shock degenerates into the Mach line $\alpha = 0$ in the limit $\epsilon \to 0$, both $\alpha/u_0\tau_0$ and $\alpha/\lambda_0\beta$ tend to zero as $\epsilon \to 0$ for a fixed β. As will be seen in §5, these quantities are indeed *uniformly* small along the nose shock. Assuming this to be the case for the moment, it follows from (3.29*a*) and (3.17*b*) that

$$\cot\delta = \lambda_0 - \epsilon^2 \frac{A+1}{4}\frac{M_{f0}^4}{\lambda_0^2}\left(\frac{\lambda_0}{2\beta}\right)^{\frac{1}{2}} e^{-\kappa\beta} F(\alpha). \tag{4.2}$$

Moreover since

$$\begin{aligned} x &= x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots \\ &= \alpha + \lambda_0\beta - \epsilon\lambda_0 R(\alpha) - \epsilon^2 F(\alpha) G(\beta), \end{aligned} \tag{4.3}$$

where

$$G(\beta) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{A+1}{2} \frac{M_{f0}^4}{(\lambda_0^3\kappa)^{\frac{1}{2}}} \operatorname{erf}(\kappa\beta)^{\frac{1}{2}}, \tag{4.4}$$

therefore

$$\frac{d\alpha}{e\beta} = \frac{\epsilon^2}{2}\frac{F(\alpha)G'(\beta)}{1 - \epsilon\lambda_0 R'(\alpha) - \epsilon^2 F(\alpha)G(\beta)}, \tag{4.5}$$

which may be readily integrated to give the equation of the nose shock in the α, β plane

$$\tfrac{1}{2}\epsilon^2 F(\alpha)^2 G(\beta) = \int_0^\alpha F(\alpha)\{1 - \epsilon\lambda_0 R'(\alpha)\} d\alpha. \tag{4.6}$$

The equation of the shock in the physical plane may be obtained by eliminating α from (4.6) and (4.3) and replacing β in these equations by r.

4.2. *Frozen limit*

In the limit $\tau_0 \to \infty$, we have $\kappa \to 0$ so that

$$G(\beta) \to \frac{(A+1)M_{f0}^4}{\lambda_0}\left(\frac{\beta}{2\lambda_0}\right)^{\frac{1}{2}}. \tag{4.7}$$

The equation of the nose shock in the α, β plane is then

$$\beta = \frac{4}{k^2\epsilon^4}\frac{\left[\int_0^\alpha F(\xi)\{1 - \epsilon\lambda_0 R'(\xi)\} d\xi\right]^2}{F(\alpha)^4}, \tag{4.8}$$

where

$$k = (A+1)M_{f0}^4/(2\lambda_0^3)^{\frac{1}{2}}. \tag{4.9}$$

The equation of the shock in the physical plane is, therefore,

$$\left.\begin{aligned} x &= \alpha + \lambda_0 r - \epsilon\lambda_0 R(\alpha) - \epsilon^2 k r^{\frac{1}{2}} F(\alpha), \\ r &= \frac{4}{k^2\epsilon^4}\frac{1}{F(\alpha)^4}\left[\int_0^\alpha F(\xi)\{1 - \epsilon\lambda_0 R'(\xi)\} d\xi\right]^2, \end{aligned}\right\} \tag{4.10}$$

which agrees with Whitham's classical formula to the lowest order of ϵ. (Note that $A = \gamma$ in the case of an ideal gas and Whitham's y is our $\alpha - \epsilon\lambda_0 R(\alpha)$.)

4.3. *The case where the relaxation decay length is of the order of a typical length of the projectile*

Let L be a typical length scale of the projectile. If the decay length κ^{-1} is of the same order as L – an important case in the theory of non-equilibrium flow – the equation of the nose shock can be simplified considerably.

Let us first non-dimensionalize all length scales by L. Specifically let us introduce new variables: $\tilde{x}$, $\tilde{r}$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{R}$, $\tilde{S}$, etc., denoting respectively x/L, r/L, α/L, β/L, R/L, S/L^2, etc. Having recast all formulas in terms of these new variables, we shall discard the tildas so that $x, r, \alpha, \beta, R, S$, etc. in this section will henceforth be understood as dimensionless. Equation (4.6), the equation of the nose shock in the α, β plane, will then assume the simple form

$$\tfrac{1}{2}\epsilon^2 F(\alpha)^2 g(\beta) = \int_0^\alpha F(\alpha)\{1-\epsilon\lambda_0 R'(\alpha)\}\,d\alpha, \tag{4.11}$$

where

$$g(\beta) = L^{\frac{1}{2}} G(\beta L) = \frac{\pi^{\frac{1}{2}}}{2} k \frac{\operatorname{erf}(\kappa L\beta)^{\frac{1}{2}}}{(\kappa L)^{\frac{1}{2}}}. \tag{4.12}$$

Likewise, its equation in the physical plane is given by (4.11) and

$$x = \alpha + \lambda_0\beta - \epsilon\lambda_0 R(\alpha) - \epsilon^2 F(\alpha) g(\beta). \tag{4.13}$$

We now assume that κL is of the order of unity. Obviously $g(\beta)$ is bounded. Consequently the integral in (4.11) is a small quantity of the order of ϵ^2 for all β's. This in turn implies that the dimensionless α is necessarily small. Now, for $\alpha \ll 1$,

$$\left.\begin{aligned} F(\alpha) &= \frac{1}{2\pi}\int_0^\alpha \frac{S''(\xi)\,d\xi}{(\alpha-\xi)^{\frac{1}{2}}} \cong \frac{S''(0)}{\pi}\alpha^{\frac{1}{2}}, \\ \int_0^\alpha F(\alpha)\{1-\epsilon\lambda_0 R'(\alpha)\}\,d\alpha &\cong \frac{2}{3}\frac{S''(0)}{\pi}\alpha^{\frac{3}{2}}\{1-\epsilon\lambda_0 R'(0)\}. \end{aligned}\right\} \tag{4.14}$$

If ϵ is taken as the tangent of the semi-nose angle of the projectile, $R'(0) = 1$. Moreover, since $S(\alpha) = \pi R(\alpha)^2$ and $R(0) = 0$, we also have $S''(0) = 2\pi$. Substituting (4.13) and (4.14) into (4.11), we have

$$\alpha = \tfrac{9}{4}\epsilon^4 g(\beta)^2 + O(\epsilon^5). \tag{4.15}$$

Substituting (4.15) into (4.13) and replacing β by r, we obtain the equation of the nose shock in the physical plane.

$$x = \lambda_0 r - \tfrac{3}{4}\epsilon^4 g(r)^2. \tag{4.16}$$

Near the tip of the projectile, $r \ll 1$ and

$$g(r) = k r^{\frac{1}{2}}\{1 - \tfrac{1}{3}\kappa L r + O(r^2)\}, \tag{4.17}$$

so that the shape of the nose shock near the tip of the projectile is

$$x = (\lambda_0 - \tfrac{3}{4}k^2\epsilon^4)r + (\tfrac{1}{2}k^2\epsilon^4\kappa L)r^2. \tag{4.18}$$

Hence,

$$(dx/dr)_{r=0} = \lambda_0 - \tfrac{3}{4}k^2\epsilon^4, \quad (d^2x/dr^2)_{r=0} = \kappa L k^2\epsilon^4, \tag{4.19}$$

at $r = 0$. Denoting the shock angle at the tip by δ_τ, one has $\cot\delta_\tau = \lambda_0 - \frac{3}{4}k^2\epsilon^4$, so that

$$\delta_\tau = \cot^{-1}\lambda_0 + \tfrac{3}{4}k^2\epsilon^4\frac{1}{1+\lambda_0^2} + \ldots = \mu_{f0} + \frac{3}{4}\frac{k^2\epsilon^4}{M_{f0}^2}. \tag{4.20}$$

The shock angle at the projectile is, therefore, independent of the relaxation time. It is indeed the shock angle for a frozen flow over the projectile (i.e. $\tau_0 \to \infty$). Moreover, the expression is in accord with that which can be deduced from Taylor–Maccoll theory. The shock curvature at the projectile is, however, affected by non-equilibrium effects. The dimensional curvature there is given by

$$\frac{1}{L}\left(\frac{d^2x}{dr^2}\right)_{r=0}\bigg/\left\{1+\left(\frac{dx}{dr}\right)^2_{r=0}\right\}^{\frac{3}{2}}$$

or

$$\frac{k^2\epsilon^4}{M_{f0}^3}\kappa = \epsilon^4\frac{\sigma k^2}{2\lambda_0 M_{f0}}\frac{1}{u_0\tau_0}. \tag{4.21}$$

Finally, for $\kappa Lr \gg 1$, the nose shock degenerates into the frozen Mach wave,

$$x = \lambda_0 r - \tfrac{3}{8}\pi\epsilon^4 k^2/\kappa L. \tag{4.22}$$

5. Verification of the *a priori* hypotheses

Two sets of hypotheses were introduced in constructing the second-order solutions: (1) that the required solution can be constructed by applying the jump conditions at $\alpha = 0$ instead of the shock wave whose position is not known until the problem is solved; (2) that $\alpha/u_0\tau_0 \ll 1$ and $\alpha/\lambda_0\beta \ll 1$ along the shock wave.

In the frozen limit where $\kappa \to 0$ or $u_0\tau_0 \to \infty$, the last hypothesis is known to be valid (Whitham 1952). If κL is of the order of unity, this can also be verified. For the nose shock is given by (4.15) and, in terms of dimensional α and β, it assumes the simple form

$$\alpha = \tfrac{9}{16}\pi(k^2\epsilon^4/\kappa)\left(\operatorname{erf}(\kappa\beta)^{\frac{1}{2}}\right)^2. \tag{5.1}$$

Recalling that both $\operatorname{erf}(\kappa\beta)^{\frac{1}{2}}$ and $(\operatorname{erf}(\kappa\beta)^{\frac{1}{2}}/(\kappa\beta)^{\frac{1}{2}})$ are bounded for all β's, we conclude immediately that $\alpha/u_0\tau_0 \ll 1$ and $\alpha/\lambda_0\beta \ll 1$, both being of the order of ϵ^4.

The validity of the first hypothesis is demonstrated if we can show that the boundary conditions at the shock wave, namely

$$p_2 = a_{f0}^2\rho_2 = -\rho_0 u_0 u_2 = (\rho_0 u_0/\lambda_0)v_2, q_2 = 0, \tag{5.2}$$

(cf. (3.9) and §3.4) are satisfied along the shock, at least to a higher order of ϵ, if not exactly.

Since $\alpha/u_0\tau_0 \ll 1$ and $\alpha/\tau_0\beta \ll 1$ at the shock, (3.29) is applicable there. The first three relationships in (5.2) are obviously satisfied everywhere along the shock. The last one, i.e. $q_2 = 0$, is only satisfied approximately with an error of $O(\epsilon^6)$. To see this, let us consider for simplicity the case where κL is of the order of unity. As in §4.3, we may simplify (3.29c) for q_2 by observing that $\alpha/L \ll 1$. Proceeding as in §4.3, one finds

$$\frac{1}{2\pi}\int_0^\alpha \frac{S'(\xi)d\xi}{(\alpha-\xi)^{\frac{1}{2}}} = \tfrac{4}{3}\alpha^{\frac{3}{2}}, \tag{5.3}$$

where use has been made of the fact that $S'(\xi) \simeq S''(0)\xi = 2\pi\xi$ for $\alpha/L \ll 1$. It follows that

$$q_2 = -\frac{\rho_0 h_{\rho 0}}{h_{q0}} \tfrac{4}{3}\sigma M_{f0}^2 \frac{\alpha}{u_0 \tau_0} \left(\frac{\alpha}{2\lambda_0 \beta}\right)^{\frac{1}{2}} e^{-\kappa\beta}. \tag{5.4}$$

Now $\alpha/u_0\tau_0$ and $\alpha/\lambda_0\beta$ are both of $O(\epsilon^4)$ along the shock, therefore $q_2 = O(\epsilon^6)$. Consequently, the second-order solution constructed under the first hypothesis does indeed satisfy approximately all the boundary conditions at the shock with an error of the order of ϵ^6.

6. Comparison with experimental results of Wegener–Klikoff

Some interesting experiments on the propagation of weak conical waves in a reactive mixture were reported in Wegener, Chu & Klikoff (1965). A supersonic projectile was fired between two metal strips on which a line of small holes was drilled parallel to the flight path. As the projectile traverses the firing range at supersonic speeds, a strong bow shock is generated which sweeps across the line of holes at supersonic speeds producing a travelling pressure disturbance over the metal strips. The increased pressure behind the shock produces a succession of gas puffs through the holes which, in turn, generate weak conical shock waves on the far side of the metal strips. The decay of these weak conical shock waves was measured. The variation of the shock strength with radial distance is expressed in curves giving the shock angle as a function of r. The model gas used in the experiments was a well understood reacting mixture

$$N_2 + N_2O_4 \rightleftarrows N_2 + 2NO_2, \tag{6.1}$$

(cf. Wegener 1961), with reactant mole fractions varying from 0 (i.e. pure nitrogen) to 0·15. Experiments conducted in pure nitrogen yield results which are indistinguishable from those obtained in air. These experimental results will now be compared with the theoretical formulas derived below.

According to (4.16), the equation of the nose shock expressed in terms of dimensional r and x is

$$x = \lambda_0 r - \tfrac{3}{4}\epsilon^4 G(r)^2, \tag{6.2}$$

where

$$G(r) = \tfrac{1}{2}\pi^{\frac{1}{2}} k \kappa^{\frac{1}{2}} \operatorname{erf}(\kappa r)^{\frac{1}{2}}. \tag{6.3}$$

The shock angle δ can then be computed from

$$\begin{aligned} \cot\delta &= dx/dr \\ &= \lambda_0 - \tfrac{3}{8}\pi^{\frac{1}{2}}\epsilon^4 k^2 e^{-\kappa r}(\operatorname{erf}(\kappa r)^{\frac{1}{2}}/(\kappa r)^{\frac{1}{2}}. \end{aligned} \tag{6.4}$$

Thus,

$$\delta = \mu_f + \tfrac{3}{8}\pi^{\frac{1}{2}}\epsilon^4 k^2 M_{f0}^{-2} e^{-\kappa r}(\operatorname{erf}(\kappa r)^{\frac{1}{2}}/(\kappa r)^{\frac{1}{2}}. \tag{6.5}$$

In terms of the shock angle $\delta(0)$ at $r = 0$, we have

$$\delta = \mu_f + \tfrac{1}{2}\pi^{\frac{1}{2}} e^{-\kappa r}(\operatorname{erf}(\kappa r)^{\frac{1}{2}}/(\kappa r)^{\frac{1}{2}})\{\delta(0) - \mu_f\}. \tag{6.6}$$

Shock angles computed from the above theoretical formula are compared with the experimentally determined shock angles in figure 1. The undisturbed conditions of the medium for the various experiment points are given in table 1. It is seen that the agreement between the experimental and theoretical values is excellent.

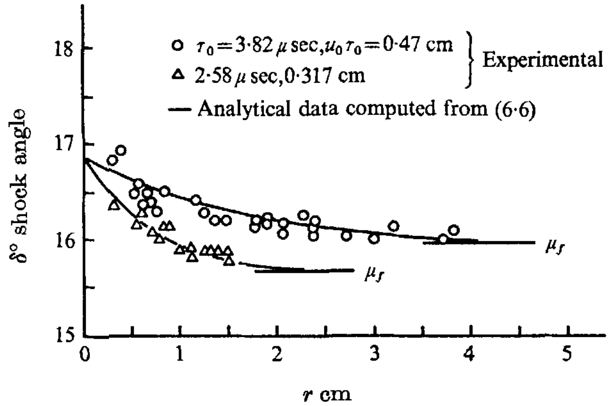

FIGURE 1. Comparison of theoretical and experimental shock wave attenuation (experimental data taken from Wegener, Chu & Klikoff 1965).

Experiment number	n_r mole Fraction	U_0 (m/sec)	M_{f0}	M_{e0}	τ_0 (sec)	$U_0\tau_0$ (cm)	T_0 (°K)	P_0 atm	μ_{f0} (degree)	σ
146	0·054	1230	3·64	3·86	2·82	0·470	0·296	1·01	15·95	0·1245
143	0·081	1230	3·70	3·96	2·58	0·317	0·297	1·00	15·68	0·1455

TABLE 1. Test condition of experiments from Wegener, Chu & Klikoff (1965)

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Appendix

One should remark that the next term in the expansion for $x(\alpha,\beta)$, cf. (4.3), is not of the order of ϵ^3 but of $\epsilon^{\frac{5}{2}}$, that is,

$$x(\alpha,\beta) = x_0+\epsilon x_1+\epsilon^2 x_2+\epsilon^{\frac{5}{2}} x_{\frac{5}{2}}+O(\epsilon^3),$$

where $x_{\frac{5}{2}}(\alpha,\beta)$ satisfies the differential equation $x_{\frac{5}{2},\beta} = 0$. The boundary condition for $x_{\frac{5}{2}}(\alpha,\beta)$ is determined by the requirement that $x = \alpha$ at $\beta = \epsilon R(\alpha)$. Thus, we find

$$x_{\frac{5}{2}} = -\frac{A+1}{2^{\frac{1}{2}}}\frac{M_{f0}^4}{\lambda_0^{\frac{3}{2}}} R(\alpha)^{\frac{1}{2}} F(\alpha).$$

Without the half power terms like $x_{\frac{5}{2}}$, the above boundary condition can never be satisfied, regardless how many more terms in the expansion is computed. To the order of $\epsilon^{\frac{5}{2}}$, the nose shock in the α,β plane assumes the form

$$\tfrac{1}{2}\epsilon^2 F(\alpha)^2 G(\beta) = \int_0^\alpha F(\alpha)\{1-\epsilon\lambda_0 R'(\alpha)-\epsilon^{\frac{5}{2}} k R(\alpha)^{\frac{1}{2}}\}\,d\alpha$$

instead of (4·6). Here, k is given by (4.9). Since $k \simeq M_{f0}^4 \lambda_0^{-\frac{3}{2}}$, the $\epsilon^{\frac{5}{2}}$ term becomes more significant in cases where M_{f0} is large. The improvement over the classical results of Whitham resulting from this term as well as terms of the order of ϵ^3 and $\epsilon^{\frac{7}{2}}$ has recently been demonstrated by Chou in a calculation which will be reported later.

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